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A NOTE ON CONTROLLABILITY OF LINEAR SYSTEMS

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Abstract

Necessary and sufficient conditions for $T - \varepsilon$ controllability of linear systems are investigated.

1. Introduction

Let X be a Banach space with norm denoted by $\|\cdot\|$, and consider the dynamical system

$$\frac{dp}{dt} = Ap + b(t)u + c(t), \tag{1.1}$$

where $p \in X$, A is a closed linear operator generating the semigroup $\{f(t)\}$, $t \geq 0$, for the class C_0 of bounded linear operator on X into itself, and $b(t)$ and $c(t)$ are continuous vector functions with values in X. The function $u(t) \in L_1[0, T]$ is called a *control*. The solution of (1.1) can be represented by the Cauchy's formula

$$\Phi(t, p, u) = f(t)p + \int_0^t f(t-\tau)b(\tau)u(\tau)d\tau + \int_0^t f(t-\tau)c(\tau)d\tau, \; 0 \leq \tau \leq T. \tag{1.2}$$

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Definition 1.1. A point p is said to be $TM - \varepsilon$ *controllable* at the point $q (T > 0,\ M > 0)$, if for arbitrary ε, there is a control $u(t) \in L_1[0, T]$, $\| u \|_{L_1} \leq M$ such that $\| \Phi(T, p, u) - q \| < \varepsilon$.

If $M = \infty$, we omit the letter M and use the term $T - \varepsilon$ controllability or ε controllability for a time T, a point p and a point q.

Definition 1.2. The dynamical system (1.2) is called T *controllable* if, for arbitrary $p, q \in X$ and arbitrary $\varepsilon > 0$, there is a control $u(t)$ such that $\| \Phi(Y, p, u) - q \| < \varepsilon$. It is called $T - \varepsilon$ *controllable* at zero if, for arbitrary $p \in X$ and $\varepsilon > 0$ there is a control $u(t)$ ensuring that $\| \Phi(T, p, u) \| < \varepsilon$.

2. Main Results

In this section we give necessary and sufficient conditions for ε controllability.

Theorem 2.1. *Let $k[0, T]$ be a linear manifold, dense in $L[0, T]$. If the point p is $TM - \varepsilon$ controllable at the point q on $L[0, T]$, then it is $TM - \varepsilon$ controllable at the point q on $k[0, T]$.*

Proof. This is obvious because $\Phi(t, p, u)$ depends continuously on u. In fact, if

$$\| u - u_0 \|_{L_1} \to 0, \quad \| \Phi(t, p, u) - \Phi(t, p, u_0) \|_X \to 0,$$

and the result follows.

We write $\pi_M(\Gamma)$, where $M > 0$, and $\Gamma = \{\gamma\}$ is an arbitrary set in X, for the closure of the set of all finite linear combinations $\sum_{i=1}^{N} \lambda_i \gamma_i$, $(\gamma_i \in \Gamma$ and the positive integer N is arbitrary) for which $\sum_{i=1}^{N} | \lambda_i | \leq M$. The smallest linear subspace containing Γ will be denoted by $\pi(\Gamma)$.

Theorem 2.2. *A point p is TM − ε controllable at a point q for* (1.1) *if and only if*

$$f(T)p - q + \int_0^T f(t-\tau)c(\tau)d\tau \in \pi_M\{f(t-\tau)b(\tau) : 0 \le \tau \le T\}. \qquad (2.1)$$

Proof of Necessity. Let $r = f(T)p - q + \int_0^T f(t-\tau)c(\tau)d\tau$. It follows from our condition and Theorem 2.1 that there is a continuous function $u(t)$ such that $\| u \|_{L_1} \le M$, and $\| \Phi(T, p, u) - q \|_X < \frac{\varepsilon}{2}$ or $\left\| r + \int_0^T f(T-\tau)b(\tau)u(\tau)dT \right\| < \frac{\varepsilon}{2}$. Corresponding to each decomposition $0 = \tau_0 < \tau_1 < \cdots < \tau_N = T$ of $[0, T]$, we choose a set of numbers $\{\theta_i\}_{i=0}^N$, $\tau_i \le \theta_i < \tau_{i+1}$, so that $\sum_{i=0}^{N-1} |u(\theta_i)| \Delta\tau_i$ is a lower Darboux sum for $\int_0^T |u(\tau)| d\tau$. Since the integrand is continuous, we can consider $\int_0^T f(T-\tau)b(\tau)u(\tau)d\tau$ as a Riemann integral and select the decomposition $\{\tau\}_{i=0}^{N-1}$ of $[0, T]$ so that

$$\left\| \sum_{i=0}^{N-1} f(T-\theta_i)b(\theta_i)u(\theta_i)\Delta\tau_i - \int_0^T f(T-\tau)b(\tau)u(\tau)d\tau \right\| < \frac{\varepsilon}{2}.$$

Thus

$$\left\| r + \sum_{i=0}^{N-1} f(T-\theta_i)b(\theta_i)u(\theta_i)\Delta\tau_i \right\| < \left\| r + \int_0^T f(T-\tau)b(\tau)u(\tau)d\tau \right\|$$

$$+ \left\| \sum_{i=0}^{N-1} f(T-\theta_i)b(\theta_i)u(\theta_i)\Delta\tau_i - \int_0^T f(T-\tau)b(\tau)u(\tau)d\tau \right\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This can be rewritten as

$$\left\| r - \sum_{i=0}^{N-1} f(T-\theta_i)b(\theta_i)[-u(\theta_i)\Delta\tau_i] \right\| < \varepsilon,$$

and we have

$$\sum_{i=0}^{N-1} | - u(\theta_i)\Delta\tau_i | \le \int_0^T | u(\tau)d\tau | \le M.$$

Hence $r \in \pi_M\{f(T-\tau)b(\tau);\ 0 \le \tau \le T\}$.

Proof of Sufficiency. If $r \in \pi_M\{f(T-\tau)b(\tau);\ 0 \le \tau \le T\}$, then there are numbers N, λ_n and t_n such that $0 \le t_1 < t_2 < \cdots < t_N \le T$, $\sum_{n=1}^{N} | \lambda_n | \le M$, and

$$\left\| r - \sum_{n=1}^{N} \lambda_n f(T - t_n)b(t_n) \right\| < \frac{\varepsilon}{2}. \tag{2.2}$$

Without loss of generality, we assume that $N \ge 2$, and $\lambda_n \ne 0$, and introduce the function $\psi(\tau, \alpha, \beta)$, $\tau \ge 0$, $0 \le \alpha < \beta$,

$$\psi(\tau, \alpha, \beta) = \begin{cases} \dfrac{1}{\beta - \alpha} & \text{if } \tau \in [\alpha, \beta], \\ 0 & \text{if } \tau \notin (\alpha, \beta). \end{cases}$$

Putting $\alpha_n = t_n$, for $n = 1, 2, \ldots, N-1$ and $\beta_N = t_N$. The β_n $(n = 1, 2, \ldots, N-1)$ and α_N can be chosen so that, for $n = 1, 2, \ldots, N$,

$$\left\| \frac{1}{\beta_n - \alpha_n} \int_{\alpha_n}^{\beta_n} f(T-\tau)b(\tau)d\tau - f(T - t_n)b(t_n) \right\| < \frac{\varepsilon}{2| \lambda_n |N}$$

(see [1, 3]), and $[\alpha_n, \beta_n] \subset [0, T]$. These inequalities can be written as

$$\left\| \int_0^T f(T-\tau)b(\tau)\psi(\tau, \alpha_n, \beta_n)d\tau - f(t - t_n)b(t_n) \right\| < \frac{\varepsilon}{2| \lambda_n |N}.$$

Let $u(\tau) = -\sum_{n=1}^{N} \lambda_n \psi(\tau, \alpha_n, \beta_n)$. Then

$$\left\| \sum_{n=1}^{N} \lambda_n f(T - t_n)b(t_n) + \int_0^T f(T-\tau)b(\tau)u(\tau)d\tau \right\|$$

$$= \left\| \sum_{n=1}^{N} \lambda_n [f_n(T - t_n) b(t_n)] - \int_0^T f(T - \tau) b(\tau) \psi(\tau, \alpha, \beta) d\tau \right\|$$

$$\leq \sum_{n=1}^{N} | \lambda_n | \frac{\varepsilon}{2 | \lambda_n | N} = \frac{\varepsilon}{2}.$$

Besides (2.2), we obtain

$$\left\| r + \int_0^T (T - \tau) b(\tau) u(\tau) d\tau \right\| < \varepsilon,$$

i.e.,

$$\| \Phi(T, p, u) - q \| < \varepsilon.$$

Moreover

$$\| u \|_{L_1} = \int_0^T | u(\tau) | d\tau \leq \sum_{n=1}^{N} | \lambda_n | \int_0^T \psi(\tau, \alpha_n, \beta_n) d\tau$$

$$= \sum_{n=1}^{N} | \lambda_n | \leq N.$$

The control $u(t)$ obtained in the proof of sufficiency is discontinuous. However, since $\overline{c[0, T]} = L_1[0, T]$, we can use Theorem 2.1 to prove the existence of a continuous control $u(t)$ such that

$$\| \bar{u} \|_{L_1} \leq M; \quad \| \Phi(T, p, \bar{u}) - q \|_X < \varepsilon.$$

Corollary 2.1. *Consider the equation*

$$\frac{dp}{dt} = Ap + bu, \tag{2.3}$$

where the vector b *is constant and* $q = 0$. *Then* (2.1) *becomes*

$$f(T) p \in \pi_M \{ f(\tau) b;\ 0 \leq \tau \leq T \}.$$

Here $\{ f(\tau) b,\ 0 \leq \tau \leq T \}$ is a finite arc of a trajectory of the point b for the equation $\frac{dp}{dt} = Ap$.

Corollary 2.2. *The point p is $T - \varepsilon$ controllable at the point q, if and only if*

$$f(\tau)p - q + \int_0^T f(T - \tau)c(\tau),\ d\tau \in \pi\{f(T - \tau)b(\tau);\ 0 \le \tau \le T\}.$$

In particular, we have the following conditions for $T - \varepsilon$ controllability of point p at zero for (2.3) under the assumptions of Corollary 2.1,

$$f(T)p \in \pi\{f(\tau)b;\ 0 \le \tau \le T\}.$$

Corollary 2.3. *For the dynamical system* (1.1) *to be $T - \varepsilon$ controllable, it is necessary and sufficient that*

$$\pi\{f(T - \tau)b(\tau);\ 0 \le \tau \le T\} = X.$$

The sufficiency is obvious, the necessity is easily proved.

If $c(t) = 0$, then the dynamical system (1.1) is $T - \varepsilon$ controllable at zero if and only if

$$\pi\{f(T - \tau)b(\tau);\ 0 \le \tau \le T\} = R[f(T)].$$

Here $R[f(T)]$ is the range of the operator $f(t)$.

References

[1] W. Arveson, Noncommutative Dynamics and E-semigroups, Springer for Maths., 2003.

[2] J. K. Hale, Theory of Functional Differential Equations, Springer-Verlag, 1977.

[3] E. B. Lee and L. Markus, Foundations of Optimal Control, SIAM Series in Appl. Math., 1968.

[4] A. M. Letov, On the theory of nonlinear control systems, Contr. Differential Equation 1(2) (1966), 139-147.

[5] W. Rudin, Real and Complex Analysis, McGraw-Hill Co., 1974.

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